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**SIMPLIFIED SWITCHING  
FUNCTIONS FOR  
TIME-OPTIMAL  
CONTROL SYSTEMS**

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## 1. INTRODUCTION

The exact solution of a time-optimal problem for a linear, normal,  $n^{\text{th}}$ -order system involves an  $(n-1)$ -dimensional surface as a switching criterion for the control variable. Even for a third-order system an engineering realization of the time-optimal controller is very complicated. An example of such a design is given by Athans and Falb (1). For more practical engineering applications simpler designs for higher order systems were developed using suboptimal controllers. In most cases the great simplification obtained for suboptimal systems well compensated for the small deviation from optimality.

The basic idea used in most suboptimal designs is that the higher order system can be approximated by a second-order system. Among others, Kalman used this approach and derived a switching curve for a third-order system (4). His method essentially consists of isolating the system modes by a coordinate transformation and then letting the two dominant roots approximately characterize the dynamic behavior of the system. Thus, Kalman implicitly assumed that the higher order system has two roots which are much more important than the rest of the roots in determining the behavior of the system. This assumption is the starting point for the method used in this thesis to analyze higher order systems.

The system under analysis contains small parameters which will reduce the order of the system to two, when they go to zero. For the degenerate second-order system sensitivity functions with respect to these small parameters can be written. Employing these sensitivity functions, a nearly time-optimal solution is found by deriving a simplified switching function.

The object of this thesis is to find a method to obtain a simplified switching function for a higher order, time-optimal system which has the properties described in the last paragraph. At the same time it will be shown how sensitivity functions can be used to find the approximate time response analytically.

## 2. ANALYSIS AND DESIGN PROCEDURE

### 2.1 Formulation of Problem

Consider the following system of differential equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = k_1 x_1 + k_2 x_2 + x_3$$

$$\lambda_1 \dot{x}_3 = -x_3 + x_4$$

$$\vdots$$

$$\lambda_m \dot{x}_n + \dots - x_n + u \tag{2-1}$$

where  $x_i$  are state variables,  $k_1$  and  $k_2$  are constants,

$\lambda_j$  are small parameters and  $u$  is the control variable.

When  $\lambda_j = 0$  for  $j = 1, 2, \dots, m$  the system (2-1) reduces to the second-order, degenerate system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = k_1 x_1 + k_2 x_2 + u \tag{2-2}$$

Any linear, second-order system with constant coefficients can be represented by equations (2-2).

For the second-order system consider the following problem. Given some initial conditions and the constraint  $u \leq 1$ , find the control,  $u(t)$ , which will transfer the state of the system to the origin of the phase plane in minimum time. The well-known solution to this problem involves a switching curve that divides the phase plane into two halves. Let this switching curve be given by

$$f_0(x_1, x_2) = 0 \quad (2-3)$$

The exact solution of the same problem for the  $n^{\text{th}}$ -order system (2-1) involves, as mentioned earlier,  $(n-1)$ -dimensional switching surfaces in the  $n$ -dimensional phase space. However, it is reasonable to think that for small  $\lambda_j$ , the solution of the  $n^{\text{th}}$ -order system should be similar to that of the second-order system. In other words for a nearly time-optimal solution for the  $n^{\text{th}}$ -order system the switching functions can be expressed in terms of a switching curve in the  $x_1x_2$ -plane. The problem is to find this switching curve.

## 2.2 Sensitivity Functions

In general the values of  $\lambda_j$  could be different as long as they are all small enough to make the exponential transients negligible. However, there is not much loss in generality by making the  $\lambda_j$ 's all equal since they are

required to be small in the first place. What "small" means quantitatively will become clear later.

The state variables,  $x_i(t, \lambda)$  for  $i = 1, 2$ , can be approximated by

$$x_i(t, \lambda) = x_i(t, 0) + \left[ \frac{\partial x_i(t, \lambda)}{\partial \lambda} \right]_{\lambda=0} \lambda \quad (2-4)$$

here the subscript of  $\lambda$  has been dropped because they are now assumed to be all equal. Let us define the sensitivity functions by

$$w_i = \left[ \frac{\partial x_i(t, \lambda)}{\partial \lambda} \right]_{\lambda=0}$$

The sensitivity functions are obtained from the following system of equations

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= k_1 w_1 + k_2 w_2 + \frac{\partial u}{\partial \lambda} \end{aligned} \quad (2-5)$$

There are two difficulties connected with the solution of these equations. First of all the function  $w_i(t)$  is not defined at  $t = 0$  in all cases but it is always defined for  $t > 0$ . For this reason special initial conditions have to be found at  $t = 0^+$  where the "+" sign indicates that  $t > 0$  by a small amount. This problem is dealt with in chapter 3.

The second difficulty occurs at the switching time,  $t_s$ , where the term  $\partial u / \partial \lambda$  is discontinuous. Consequently there will be a "jump" in  $w_2(t)$  at  $t = t_s$ . Let us denote this "jump" by  $\Delta w_2$ . It will also be shown in chapter 3 how

$$\Delta w_2 = w_2(t_s^+) - w_2(t_s^-) \quad (2-6)$$

can be computed.

Now it is possible to write the sensitivity functions for all  $t > 0$  and  $t \neq t_s$ . Using equation (2-4), approximate trajectories of the  $n^{\text{th}}$ -order system can be found.

### 2.3 General Form of Switching Function

Assume that the switching curve for the  $n^{\text{th}}$ -order system is defined by

$$f(x_1, x_2, \lambda) = 0$$

This function is unknown and has to be found in some way. An approximate expression of  $f(x_1, x_2, \lambda)$  in terms of the switching function for the second-order system is given by

$$f(x_1, x_2, \lambda) = f(x_1, x_2, 0) + \left[ \frac{\partial f(x_1, x_2, \lambda)}{\partial \lambda} \right]_{\lambda=0} \lambda \quad (2-7)$$

In order to simplify the notation, let us write

$$f = f(x_1, x_2, \lambda)$$

$$f_0 = f(x_1, x_2, 0)$$

Equation (2-7) can be written as

$$f = f_0 + \left[ \frac{\partial f}{\partial x_1} \right]_{\lambda=0} w_1 \lambda + \left[ \frac{\partial f}{\partial x_2} \right]_{\lambda=0} w_2 \lambda \quad (2-8)$$

Now if we make the following assumption

$$\left[ \frac{\partial f}{\partial x_i} \right]_{\lambda=0} = \frac{\partial f_0}{\partial x_i} \quad i = 1, 2$$

equation (2-8) becomes

$$f = f_0 + \frac{\partial f_0}{\partial x_1} w_1 \lambda + \frac{\partial f_0}{\partial x_2} w_2 \lambda \quad (2-9)$$

Hence, the switching function for the  $n^{\text{th}}$ -order system can be expressed in the form

$$f = f_0 + f_1$$

where

$$f_1 = \frac{\partial f_0}{\partial x_1} w_1 \lambda + \frac{\partial f_0}{\partial x_2} w_2 \lambda \quad (2-10)$$

Remember that  $f_0$  is the switching function for the second-order system and  $f_1$  accounts for the shift of this switching function by going to the  $n^{\text{th}}$ -order system.

For the systems considered in this thesis, it is possible to express the sensitivity functions as a linear function of the  $x_i$ 's of the second-order systems. In general

$$w_1 = a_1x_1 + a_2x_2 + a_3$$

$$w_2 = b_1x_1 + b_2x_2 + b_3 \quad (2-11)$$

where the a's and b's are constants. These constants depend on initial conditions in most cases. Since a switching curve should be independent of initial conditions, it is not meaningful to use the constants of (2-11) in the switching function. The usefulness of equations (2-11) lies in the fact that they give the correct functional dependence on the state variables,  $x_i$ , for the approximate switching curve (2-9) regardless of the initial conditions.

From (2-9) and (2-11) the switching function for the  $n^{\text{th}}$ -order system is thus determined within some undefined constants. These unknown constants will be computed by some other means. But first let us consider the problem of finding the special initial conditions for the sensitivity functions.

### 3. SPECIAL INITIAL CONDITIONS FOR SENSITIVITY FUNCTIONS

#### 3.1 Chang's Method

It was mentioned that a difficulty arises in finding the initial conditions for the sensitivity equations (2-5). The function  $w_1(t)$  may not be continuous at  $t = 0$ . To illustrate this point consider, the simple system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \lambda \dot{x}_2 &= -x_2 + u\end{aligned}\tag{3-1}$$

with zero initial conditions and  $u$  as a step function. When  $\lambda = 0$ , the sensitivity equation is

$$\dot{w}_1 = 0\tag{3-2}$$

The usual initial conditions for sensitivity functions with respect to variable parameters are zero. But this will make  $w_1(t) = 0$  which is obviously wrong. The trouble is that (3-2) is not defined at  $t = 0$ . One way to avoid this difficulty is to use  $w_1(0^+)$  as an initial condition for (3-2).

A direct solution of (3-1) gives

$$x_1 = ut + u\lambda(e^{-t/\lambda} - 1)\tag{3-3}$$

Differentiating (3-3) with respect to  $\lambda$  and then setting  $\lambda = 0$ , we have

$$\left[ \frac{\partial x_1}{\partial \lambda} \right]_{\lambda=0} = w_1(t) = -u \quad (3-4)$$

Therefore a correct initial condition for (3-2) is

$$w_1(0^+) = -u.$$

Chang developed a method to find the initial conditions,  $w_1(0^+)$ , (2). For the system (2-1) this method applies in the following way. First the Laplace transforms are written for all equations in (2-1). All initial conditions are equal to zero except for  $x_1(0)$  and  $x_2(0)$ . But  $x_1(0)$  and  $x_2(0)$  can also be set to zero because they do not affect the end result. Solving the transform equations for  $X_1(s, \lambda)$  and  $X_2(s, \lambda)$  gives

$$X_1(s, \lambda) = \frac{u(0)}{s(s^2 - k_2 s - k_1) (\lambda_1 s + 1) \cdots (\lambda_m s + 1)} \quad (3-5a)$$

$$X_2(s, \lambda) = \frac{u(0)}{(s^2 - k_2 s - k_1) (\lambda_1 s + 1) \cdots (\lambda_m s + 1)} \quad (3-5b)$$

Remember that  $\lambda = \lambda_j$  for  $j = 1, 2, \dots, m$ . Differentiating (3-5a) with respect to  $\lambda$  and then letting  $\lambda = 0$ , we get

$$\left[ \frac{\partial X_1(s, \lambda)}{\partial \lambda} \right]_{\lambda=0} = W_1(s) = \frac{-\mu u(0)}{s^2 - k_2 s - k_1}$$

The initial-value theorem gives

$$\begin{aligned} w_1(0^+) &= \lim_{s \rightarrow \infty} sW_1(s) \\ &= 0 \end{aligned}$$

Now differentiating (3-5b) with respect to  $\lambda$  and setting  $\lambda = 0$  gives

$$\left[ \frac{\partial X_2(s, \lambda)}{\partial \lambda} \right]_{\lambda=0} = W_2(s) = \frac{-\mu(0)s}{s^2 - k_2s - k_1}$$

Again from the initial-value theorem

$$\begin{aligned} w_2(0^+) &= \lim_{s \rightarrow \infty} sW_2(s) \\ &= -\mu(0). \end{aligned}$$

In summary the initial conditions for the sensitivity functions defined by the equations (2-5) are

$$w_1(0^+) = 0$$

$$w_2(0^+) = -\mu(0) \quad (3-6)$$

### 3.2 Kokotovic'-Rutman Structural Method

An interesting alternative approach to determine the initial conditions of the sensitivity functions is given by Kokotovic' and Rutman (5). The method consists of drawing a block diagram of the original system and the sensitivity system for  $\lambda > 0$  and then manipulating the blocks until it is possible to let  $\lambda = 0$ .

To illustrate this method, let us find the initial conditions of the sensitivity functions for the following system which will be used in an example later.

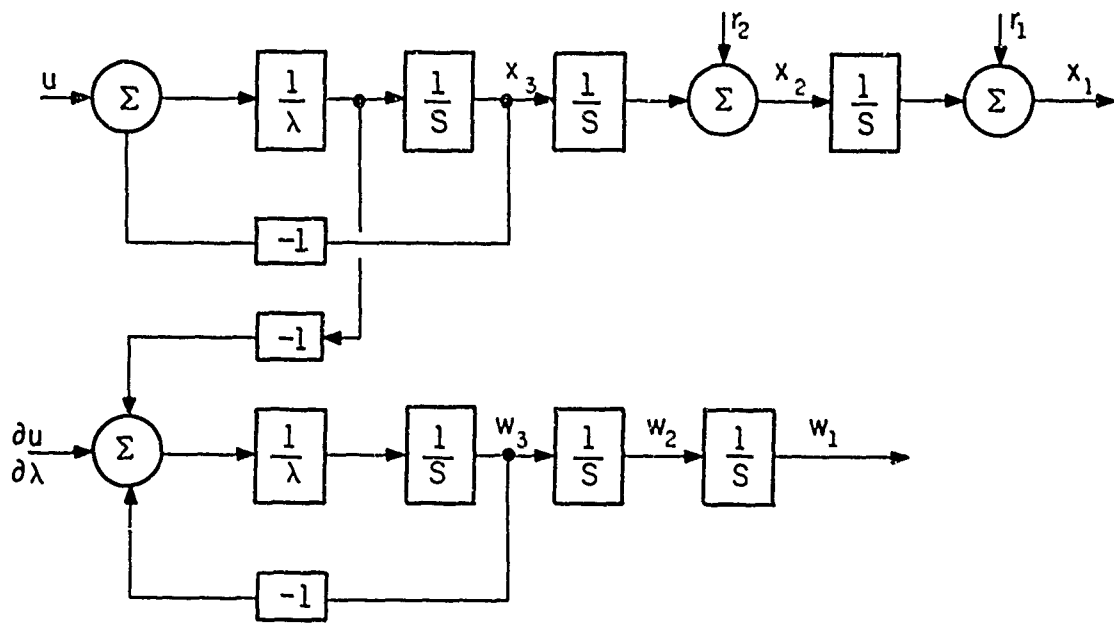
$$\begin{aligned} \dot{x}_1 &= x_2 & x_1(0) &= r_1 \\ \dot{x}_2 &= x_3 & x_2(0) &= r_2 \\ \dot{x}_3 &= -x_3 + u & x_3(0) &= 0 \end{aligned} \quad (3-7)$$

The corresponding system of sensitivity equations is of the form

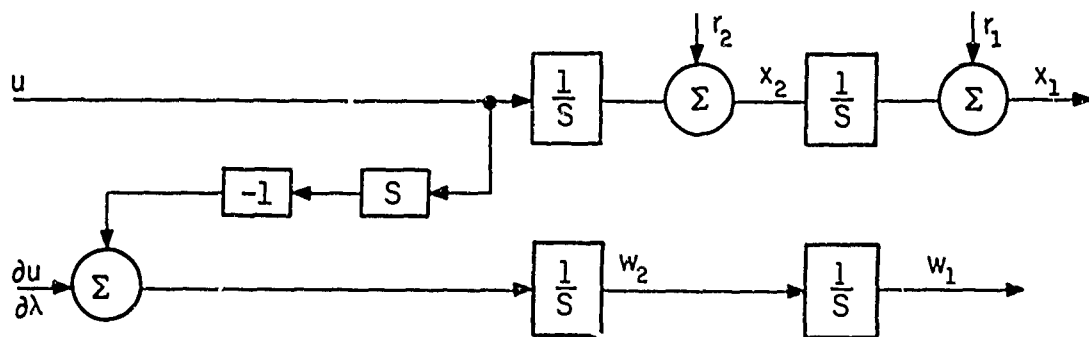
$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= w_3 \\ \dot{w}_3 &= -w_3 - \dot{x}_3 + \frac{\partial u}{\partial \lambda} \end{aligned} \quad (3-8)$$

In Figure 1a the original system diagram and the sensitivity model are shown. Figure 1b is obtained after two transformations. First the point of application for  $x_3$  is moved to the other side of the integrator and a differential operator is added in the connecting branch of the two models. Now it is possible to let  $\lambda = 0$ . The transfer function from  $u$  to  $x_3$  as  $\lambda$  goes to zero is

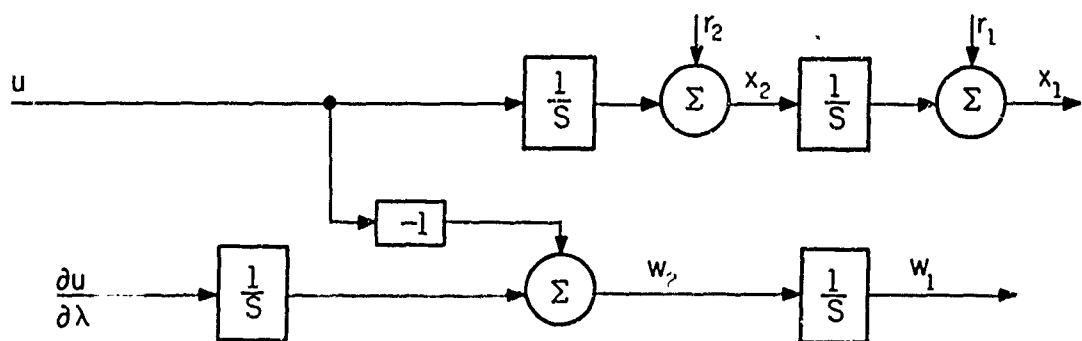
$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda s + 1} = 1$$



(a)



(b)

Figure 1. Determination of Initial Conditions for  $w_1$

The final diagram, Figure 1c has the differential operator removed by moving the point of application to the other side of the integrator in the sensitivity model.

From Figure 1c the initial conditions can be written by inspection.

$$w_1(0^+) = 0$$

$$w_2(0^+) = -u(0)$$

### 3.3 Discontinuity at Switching Time

The function  $w_2(t)$  is discontinuous at  $t=t_s$  because the control variable,  $u(t)$ , is switching from one limiting value to the other one instantaneously. As mentioned previously there will be a "jump" in  $w_2(t)$  at  $t=t_s$  and it becomes necessary to find

$$w_2(t_s^+) = w_2(t_s^-) + \Delta w_2 \quad (3-9)$$

in order to solve the sensitivity equations.

If it is possible to find a transformation of the system (2-1) such that the transformed system is a function of  $t-t_s$  but otherwise has the same form and initial conditions as the original system, the condition (3-9) can be determined from the initial conditions of the transformed system. First let us show that such a transformation exists for the system (2-1).

The system (2-1) can be expressed in matrix form as shown below.

$$\dot{\underline{x}}(t) = A \underline{x}(t) + \underline{z} \quad (3-10)$$

when  $\underline{x}$  and  $\underline{z}$  are  $n$ -vectors and  $A$  is an  $(n \times n)$  matrix. The matrix  $A$  is given by

$$A = \begin{bmatrix} 0 & 1 & & & & \\ k_1 & k_2 & 1 & & & 0 \\ & 0 & -\frac{1}{\lambda} & 1 & & \\ & & \text{---} & \text{---} & \text{---} & \\ 0 & & & 0 & -\frac{1}{\lambda} & 1 \\ & 0 & & & -\frac{1}{\lambda} & \\ & & & & 0 & -\frac{1}{\lambda} \end{bmatrix} \quad (3-11)$$

The vector  $\underline{z}$  has only one non-zero element in the  $n^{\text{th}}$  row which is  $u/\lambda$ .

Let us define the new time variable,  $T$ , as

$$T = t - t_s \quad (3-12)$$

The initial conditions for (3-10) are assumed to be zero. Suppose now that there exists the following matrix equation

$$\dot{\underline{y}}(T) = A \underline{y}(T) + \underline{z} \quad (3-13)$$

with the initial conditions  $\underline{y}(0) = \underline{0}$ . The next step is to

find the transformation of  $\underline{x}$  which will give  $\underline{y}$  in (3-13).

The general solutions of (3-10) and (3-13) are

$$\underline{x}(t) = \int_0^t e^{A(t-p)} \underline{z} dp \quad (3-14)$$

$$\underline{y}(T) = \int_0^T e^{A(T-p)} \underline{z} dp \quad (3-15)$$

In the above equations  $p$  is a dummy variable. Substituting  $t = t_s + T$  into (3-14) gives

$$\begin{aligned} \underline{x}(T) &= \int_0^{T+t_s} e^{A(t_s+T-p)} \underline{z} dp \\ &= e^{At_s} \underline{y}(T) + \int_T^{T+t_s} e^{A(T-p)} \underline{z} dp \end{aligned} \quad (3-16)$$

Solving (3-16) for  $\underline{y}(T)$ , we have

$$\underline{y}(T) = e^{-At_s} \underline{x}(t) + \int_{T+t_s}^T e^{A(T-p)} \underline{z} dp \quad (3-17)$$

Equation (3-17) gives the transformation from  $\underline{x}(t)$  to  $\underline{y}(T)$ , so that  $\underline{y}(T)$  satisfies the equation (3-13). As long as the state transition matrix exists, it is possible to find this transformation.

For the system (3-13) the sensitivity functions and their initial conditions can be determined in the same way as it has been done for the system (3-10). Only different symbols are used. Let us denote the new sensitivity functions by

$$v_i(T) = \left[ \frac{\partial y_i(T, \lambda)}{\partial \lambda} \right]_{\lambda = 0} \quad i = 1, 2$$

The initial conditions are

$$v_1(T=0^+) = 0$$

$$v_2(T=0^+) = -\mu(T=0^+) \quad (3-18)$$

The relation between  $v_i(T)$  and  $w_i(t)$  can be found from (3-16) in the following manner. For the second-order system (2-2) the matrix  $A$  is given by

$$A = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} \quad (3-19)$$

and the vector  $\underline{z}$  is

$$\underline{z} = \begin{bmatrix} 0 \\ u \end{bmatrix} \quad (3-20)$$

Substituting (3-19) and (3-20) into (3-16), a relation between  $\underline{y}(T)$  and  $\underline{x}(t)$  is obtained. Differentiating this

relation with respect to  $\lambda$ , we get the equations relating  $V_1(T)$  and  $w_1(t)$ .

Consider the particular case when  $k_1 = k_2 = 0$ . The solution of equation (3-16) for this case is

$$\begin{aligned} X_1(t) &= y_1(T) + t_s y_2(T) + \frac{ut_s^2}{2} \\ X_2(t) &= y_2(T) + ut_s \end{aligned} \quad (3-21)$$

In equations (3-21)  $u$  is considered to be constant and continuous. The fact that it is actually discontinuous at the switching time is accounted for by the special initial conditions of the transformed system (3-13). With this in mind (3-21) can be differentiated with respect to  $\lambda$  and the result is

$$\begin{aligned} w_1(t) &= V_1(T) + t_s V_2(T) \\ w_2(t) &= V_2(T) \end{aligned} \quad (3-22)$$

Evaluating (3-22) at  $t = t_s^+$  or  $T = 0^+$  and substituting (3-18) gives

$$\begin{aligned} w_1(t_s^+) &= -t_s \mu(t_s^+) \\ w_2(t_s^+) &= -\mu(t_s^+) \end{aligned}$$

Since  $u(t_s^+) = -u(t=0)$  equations (3-23) become

$$w_1(t_s^+) = t_s \mu(t=0)$$

$$w_2(t_s^+) = \mu(t=0) \quad (3-24)$$

For this particular system the "jump" in  $w_2$  at the switching time is

$$\Delta w_2 = - 2\mu(t=0) \quad (3-25)$$

In a similar way the "jump" conditions can be derived for other cases of the general system (2-1).

#### 4. SWITCHING FUNCTION FOR A PARTICULAR SYSTEM

##### 4.1 Functional Form

The general form of the system under consideration is given by (2-1). When  $k_1 = k_2 = 0$ , we have the following system

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= x_3 \\
 \lambda \dot{x}_3 &= -x_3 + u \\
 &\vdots \\
 \lambda \dot{x}_n &= -x_n + u
 \end{aligned} \tag{4-1}$$

The corresponding second-order system for  $\lambda = 0$  is

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= u
 \end{aligned} \tag{4-2}$$

Let the initial conditions be denoted by

$$\begin{aligned}
 x_1(0) &= r_1 \\
 x_2(0) &= r_2
 \end{aligned} \tag{4-3}$$

The solution of (4-2) is

$$\begin{aligned}
 x_1 &= \frac{ut^2}{2} + r_2 t + r_1 \\
 x_2 &= ut + r_2
 \end{aligned}
 \tag{4-4}$$

The sensitivity equations for the system (4-2) are

$$\begin{aligned}
 \dot{w}_1 &= w_2 \\
 \dot{w}_2 &= \frac{\partial u}{\partial \lambda}
 \end{aligned}
 \tag{4-5}$$

with the initial conditions (3-6). The solution of (4-5) is for  $0 \leq t < t_s$

$$\begin{aligned}
 w_1 &= -\mu u(0)t \\
 w_2 &= -\mu u(0)
 \end{aligned}
 \tag{4-6}$$

Comparing equations (4-4) and (4-5) one can see that they are related in the following way

$$\begin{aligned}
 w_1 &= a_2 x_2 + a_3 \\
 w_2 &= b_3
 \end{aligned}
 \tag{4-7}$$

The constants in the above equations will vary with the initial conditions. But the important point is that the form of the equations (4-7) is uniquely determined for all initial conditions.

The optimal switching function for the system (4-2) is

$$f_0 = x_1 + \frac{1}{2} x_2 |x_2| \tag{4-8}$$

Differentiating (4-8) with respect to  $x_1$  and  $x_2$  gives

$$\frac{\partial f_0}{\partial x_1} = 1$$

$$\frac{\partial f_0}{\partial x_2} = |x_2| \quad (4-9)$$

If we substitute (4-7) and (4-9) into (2-10) we get

$$f_1 = (a_2 x_2 + a_3) \lambda + |x_2| b_3 \lambda \quad (4-10)$$

Defining new constants (4-10) can be written as

$$f_1 = b \lambda x_2 + c \lambda \quad (4-11)$$

where

$$b = a_2 + b_3 \frac{x_2}{|x_2|}$$

$$c = a_3$$

Finally the total switching function from (4-8) and (4-11) is

$$f = x_1 + \frac{1}{2} x_2 |x_2| + b \lambda x_2 + c \lambda \quad (4-12)$$

The next task is to determine the two constants  $b$  and  $c$  uniquely and independent of the initial conditions. This problem is solved in the next section.

The procedure outlined above can easily be applied to two other cases of the general system (2-1): 1.) when

$k_1 + 0$  and  $k_2 \neq 0$  2.)  $k_1 \neq 0$  and  $k_2 \neq 0$  and the eigenvalues of (2-2) are distinct and real. For the case of complex eigenvalues difficulties arise because the optimal switching curve cannot be expressed in a nice analytical form. However, if the optimal switching curve is approximated by some analytical function of the state variables, it is possible to treat this case in the same way.

#### 4.2 Determination of Constants

In order to compute the constants  $b$  and  $c$ , it is necessary to derive the equations for the approximate trajectory of the  $n^{\text{th}}$ -order system as given by (2-4). In this derivation the following initial condition will be used.

$$\begin{aligned} x_1(0) &= r_1 \\ x_2(0) &= 0 \end{aligned} \tag{4-13}$$

The initial conditions (4-3) could also be used. Since the final result is independent of the initial conditions, it is more convenient to use (4-13). The mathematical analysis is simpler with (4-13). However, the initial conditions have to be general enough, so that every point on the switching curve can be reached through them. This requirement is satisfied by (4-13).

Another simplification of the analysis is obtained by only considering one half of the switching curve. The other

half is determined by symmetry. By restricting  $r_1$  to positive values, the trajectories will only reach the lower half of the switching curve.

For this case the optimal control sequence for the system (4-2) is  $u = (-1, +1)$ . The solution is

For  $0 \leq t \leq t_s$ :

$$x_1 = -\frac{t^2}{2} + r_1$$

$$x_2 = -t$$

(4-14)

For  $t_s \leq t \leq t_f$ :

$$x_1 = \frac{t^2}{2} - 2t_s t + t_s^2 + r_1$$

$$x_2 = t - 2t_s$$

(4-15)

The solution of the corresponding sensitivity equations (4-5) with the initial conditions (3-6) and the "jump" condition (3-25) is

For  $0 \leq t < t_s$ :

$$w_1 = mt$$

$$w_2 = m$$

(4-16)

For  $t_s < t \leq t_f$ :

$$w_1 = -mt + 2mt_s$$

$$w_2 = -m \quad (4-17)$$

The approximate equations for the  $n^{\text{th}}$ -order system (4-1) are

For  $0 \leq t < t_s$ :

$$\begin{aligned} x_1 &= -\frac{t^2}{2} + m\lambda t + r_1 \\ x_2 &= -t + m\lambda \end{aligned} \quad (4-18)$$

For  $t_s < t \leq t_f$ :

$$\begin{aligned} x_1 &= \frac{t^2}{2} - (2t_s + m\lambda)t + t_s^2 + 2m\lambda t_s + r_1 \\ x_2 &= t - 2t_s - m\lambda \end{aligned} \quad (4-19)$$

An expression for the switching time,  $t_s$ , is obtained from the intersection of the switching curve (4-12) and the trajectory defined by equations (4-18). The final result is

$$t_s = \frac{1}{2}(b-2)m\lambda + \left[ r_1 + \frac{1}{4}(b^2+2)m^2\lambda^2 + cm\lambda \right]^{\frac{1}{2}} \quad (4-20)$$

A condition that has to be satisfied by the trajectory (4-19) is that it should reach the origin at the final time,  $t_f$ . In other words,  $x_1(t_f) = x_2(t_f) = 0$ . From equations (4-19) this will only happen if

$$t_f = 2t_s + m\lambda \quad (4-21)$$

and

$$-\frac{1}{2} t_f^2 + t_s^2 + 2m\lambda t_s + r_1 = 0 \quad (4-22)$$

Substituting (4-21) into (4-22) and solving for  $t_s$  gives

$$t_s = \left[ r_1 - \frac{m^2 \lambda^2}{2} \right]^{\frac{1}{2}} \quad (4-23)$$

The two expressions for  $t_s$ , (4-20) and (4-23), are equal if

$$\begin{aligned} b &= 2 \\ c &= -2m\lambda \end{aligned} \quad (4-24)$$

It turns out that this is the only choice of the constants  $b$  and  $c$  which is independent of  $r_1$  and at the same time makes equation (4-20) equal to (4-23).

Putting the constants (4-24) into the switching function (4-12) and correcting for the sign gives

$$f = x_1 + \frac{1}{2} x_2 |x_2| + 2m\lambda x_2 + 2m\lambda^2 \frac{x_2}{|x_2|} \quad (4-25)$$

Remember that  $m = n-2$ . So for a third-order system  $m = 1$ . The switching function for that case is almost in perfect agreement with one derived by Kalman for the same system by a completely different method (4). The only difference is that in (4-25) the constant multiplier of  $\lambda^2$  is 2 while in Kalman's equation it is 1/2.

Figure 2 shows two approximate switching curves for a third-and-fourth-order system when  $\lambda = 1/2$ . The optimal switching curve for the second-order system is also shown for comparison. Note that the two approximate switching curves do not go exactly through the origin. The "miss" at the origin is proportional to  $\lambda^2$ . If  $\lambda$  is kept smaller than one, the curves will come quite close to the origin.

#### 4.3 Numerical Examples

In this section the time response of the third-order system (3-7) will be computed for two different initial conditions and  $\lambda = 0.5$  using the derived switching function (4-25).

First let us choose  $r_1 = 4.125$  and  $r_2 = 0$ . Consequently the switching time,  $t_s$ , is 2 and the final time,  $t_f$ , is 4.5. The equations describing the trajectory are given below.

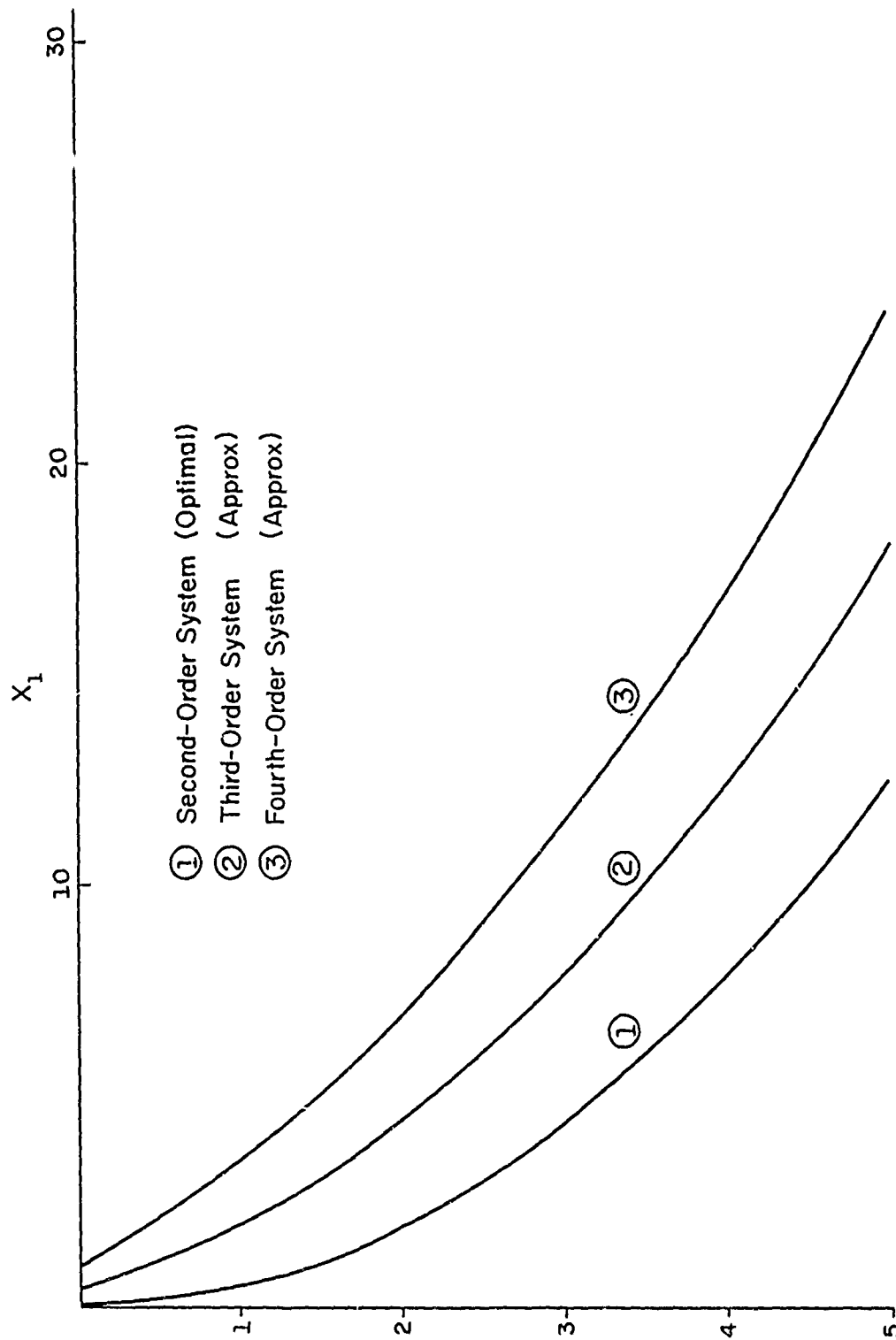
For  $0 \leq t < t_s$ :

$$w_1 = t$$

$$w_2 = 1$$

$$x_1 = -0.5t^2 + 4.125$$

$$x_2 = -t$$



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Figure 2. Switching Curves

$$x_1 + 0.5w_1 = -0.5t^2 + 0.5t + 4.125$$

$$x_2 + 0.5w_2 = -t + 0.5$$

For  $t_s < t \leq t_f$ :

$$w_1 = -t + 4$$

$$w_2 = -1$$

$$x_1 = 0.5t^2 - 4t + 8.125$$

$$x_2 = t - 4$$

$$x_1 + 0.5w_1 = 0.5t^2 - 4.5t + 10.125$$

$$x_2 + 0.5w_2 = t - 4.5$$

Let the exact solution of the system of differential equations with  $u$  switching from  $-1$  to  $+1$  at  $t = t_s$  be denoted by  $x_i^*$  for  $i = 1, 2$ . Figure 3 shows the various curves for this example. Note that there is an error of about  $\lambda^2$  between the exact and approximate curve of  $x_1$  at  $t = t_f$ . Since the optimal time for the second-order system when  $\lambda = 0$  is 4.06, one can see that the time response is very nearly time-optimal.

As a second example let us use the initial conditions  $r_1 = 0$  and  $r_2 = 2$ . The constants for the switching curve were derived with zero initial conditions for  $x_2$ . This

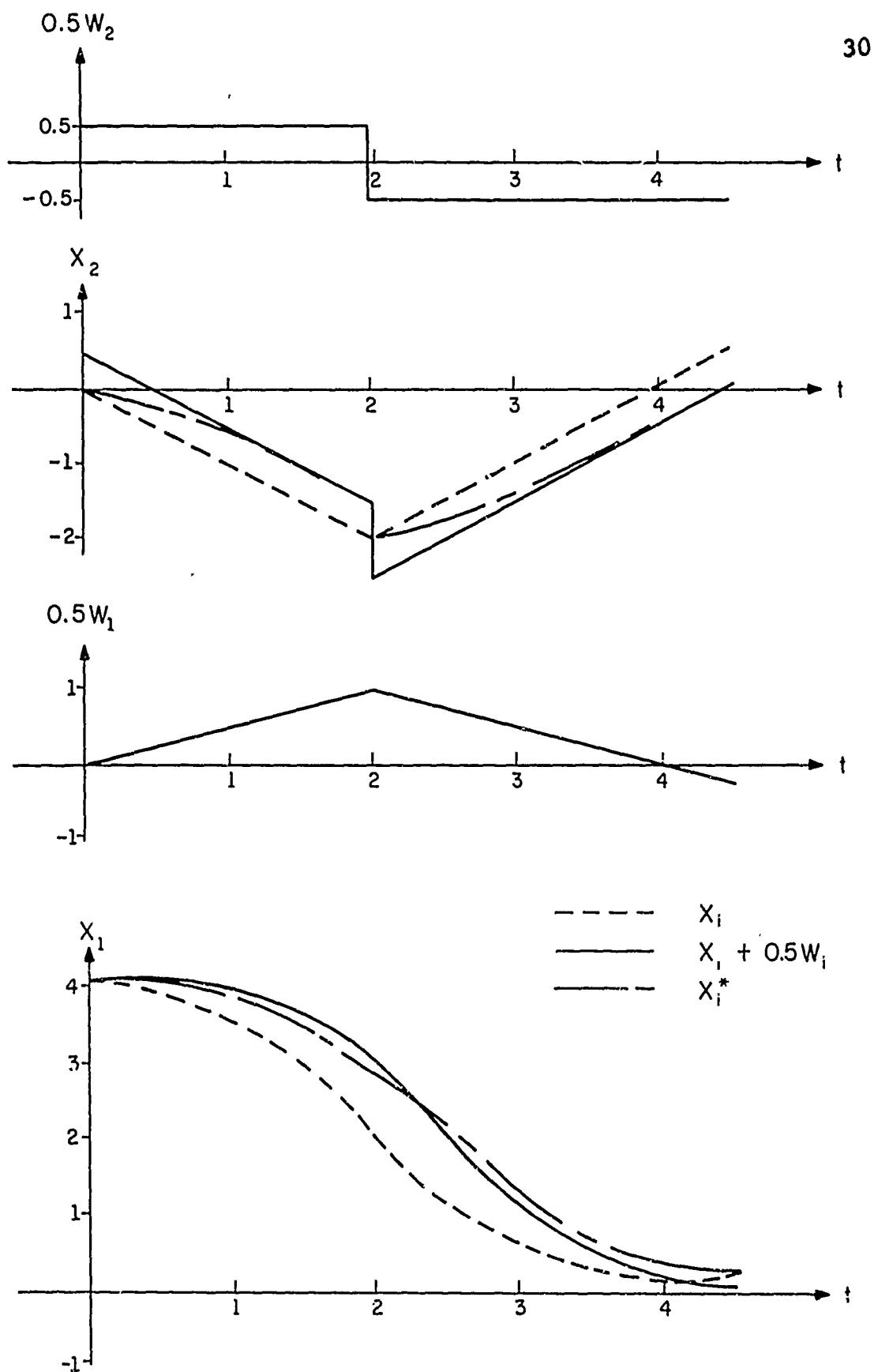


Figure 3. Time Response for Example 1

example will illustrate the fact that the switching curve nevertheless is valid for non-zero initial conditions of  $x_2$ .

The formulas for  $t_s$  and  $t_f$  are easily derived in the same way as before. The result is

$$t_s = 1.707r_2 + 0.707$$

$$t_f = 2.414 (r_2 + \lambda)$$

For this example  $t_s = 3.768$  and  $t_f = 6.035$ .

The trajectory is described by the following equations.

For  $0 \leq t < t_s$ :

$$w_1 = t$$

$$w_2 = 1$$

$$x_1 = -0.5t^2 + 2t$$

$$x_2 = -t + 2$$

$$x_1 + 0.5w_1 = -0.5t^2 + 2.5t$$

$$x_2 + 0.5w_2 = -t + 2.5$$

For  $t_s < t \leq t_f$ :

$$w_1 = -t + 7.536$$

$$w_2 = -1$$

$$x_1 = 0.5t^2 - 5.535t + 14.20$$

$$x_2 = t - 5.535$$

$$x_1 + 0.5w_1 = 0.5t^2 - 6.035t + 17.97$$

$$x_2 + 0.5w_2 = t - 6.035$$

Figure 4 shows the various curves for this example. Again the time response is a good approximation of the time-optimal solution.

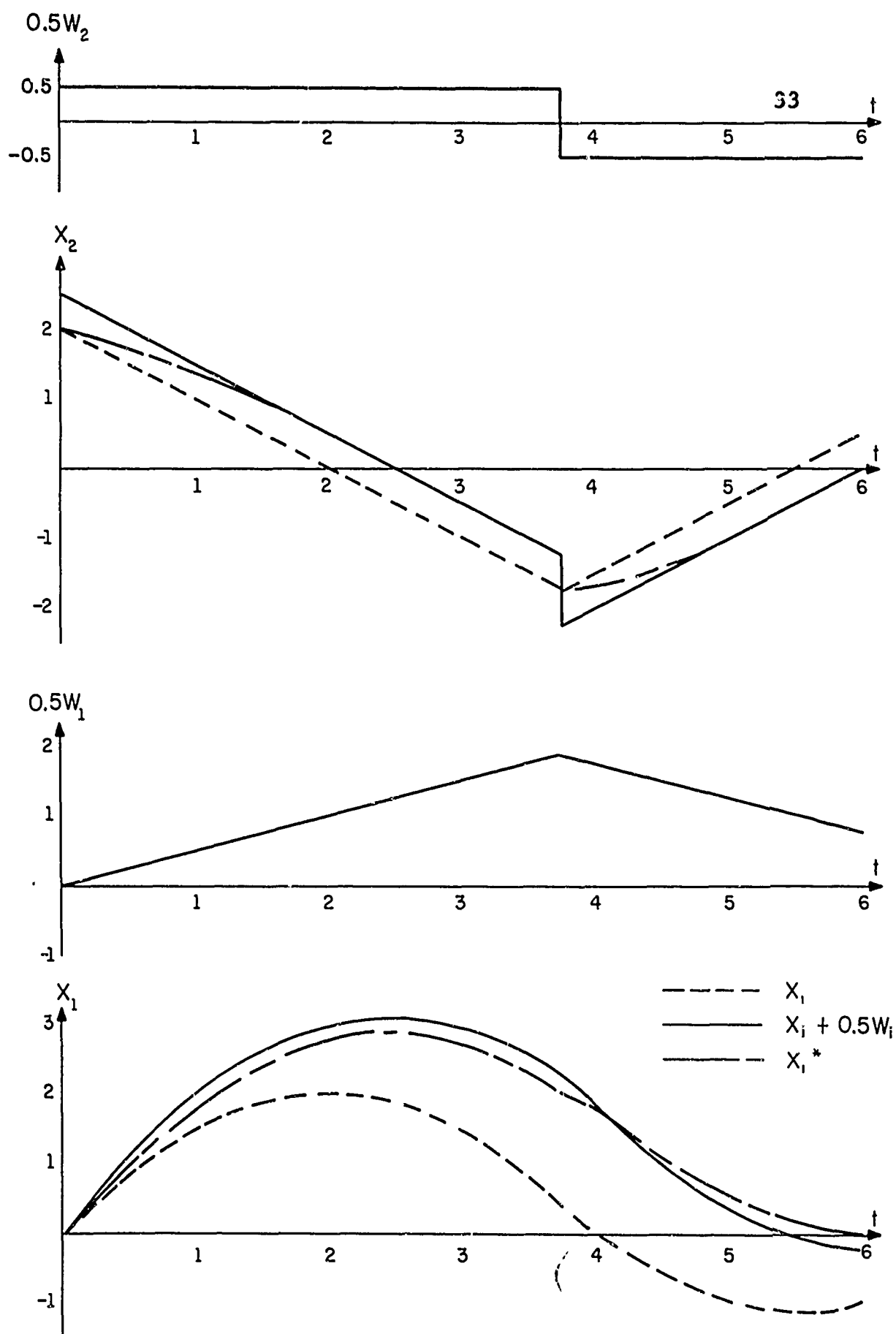


Figure 4. Time Response for Example 2

## 5. SUMMARY AND CONCLUSION.

A method has been presented to derive simplified switching functions for the design of third or higher order, time-optimal control systems. The type of system, that has been considered, is characterized by small parameters which increase the order of the system. The sensitivity functions with respect to these small parameters have been the main tool of analysis. Two existing methods have been used to determine special initial conditions for the sensitivity equations because these equations are discontinuous at  $t = 0$ . It has been shown that the same methods can be applied to deal with the discontinuity of the sensitivity functions at the switching time by a transformation of the original system.

A particular system has been analyzed in more detail. The degenerate, second-order system for this case is the well-known double integral plant. The complete switching curve has been determined for this system. Using this switching curve the approximate time response was computed with the aid of sensitivity functions. The results show that the response is nearly time-optimal.

For a system characterized by a second-order system, which is more complicated than the double integral plant, it might be difficult to analytically determine unique constants in the switching function. In that case perhaps the

best constants can be found experimentally. As already indicated, complex roots of the second-order system also present special problems. All these cases need further study, for which there was no time in connection with this thesis unfortunately.

This investigation above all demonstrates how sensitivity functions with respect to small parameters, that change the system order, can be used to greatly simplify the analysis of higher order systems.

## LIST OF REFERENCES

1. Athans, M. and Falb, P.L., Optimal Control, McGraw-Hill, Inc., 1966.
2. Chang, S. S. L., Synthesis of Optimum Control Systems, McGraw-Hill, Inc., 1961.
3. DeRusso, P. M., Roy, R. J., and Close, M. C., State Variables for Engineers, John Wiley and Sons, Inc., 1965.
4. Kalman, R. E., "Analysis and Design Principles of Second and Higher Order Saturating Seiromechanisms", Transactions of the American Institute of Electrical Engineers, Part II, Applications and Industry, Vol. 74, November, 1955, pp. 294-310.
5. Radanovic', L., Sensitivity Methods in Control Theory, Pergamon Press, 1964.

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